Between Mathematics and Natural Philosophy: Towards a Social History of Infinitesimals c. 1700

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Steven Shapin's influential book, *A Social History of Truth* (1994), forcefully articulated the idea that epistemic values were culturally embedded and socially constructed. Shapin showed that what counted as truth for Robert Boyle and most of his fellow members of the Royal Society was shaped by English early modern gentlemanly culture—prominently including notions of decorum and proper rules of conduct. Which methods were appropriate to settle disputes in natural philosophy was also decided by taking into account rules and conventions of gentlemanly behaviour.[1]

Interestingly, mathematics was left out of the picture. Or, to be more precise, Shapin used mathematics as the perfect foil to highlight the novel features of gentlemanly experimental philosophy. He contrasted the new and growing community of experimental actors courteously exchanging views on matters of fact in public spaces, to small mathematical coteries, access to which was restricted both by technicalities and, even more important, by abstruse specialized (and therefore not gentlemanly) language. Disagreements in experimental philosophy, which were to be expected, would be solved with gentlemanly decorum through the "free action of testifying members", which Shapin opposed to "mathematical expectations of precision" and the "iron chains of logic" in mathematical demonstrations.[2] In Shapin's book not only mathematics was left out of culturally shaped experimental philosophy,

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[1] S. Shapin, *A Social History of Truth. Civility and Science in Seventeenth-Century England* (Chicago, 1994).

[2] Shapin, *A Social History of Truth*, p. 352.

but the book reinforced the widespread, common perception of mathematics as an ideal realm where some transcendent logic rules.

For obvious reasons mathematics is a notoriously difficult case for epistemological social constructivism. The prevailing image of mathematics takes for granted that mathematical argument is essentially self-referential in two fundamental senses. First in the sense that the truth of a mathematical result or proposition does not depend on its agreement with something external to the mathematical discourse, but on its agreement with other mathematical results. Second, in that mathematical objects are clear and well-defined ideas that entail no obvious contradiction or inconsistency. That is what makes mathematics special and what suggests a historical continuity and even permanence in the principles and methods of mathematical thought—the "iron chains of logic" that Shapin used to characterize mathematics.

However, a long tradition in the historiography recognizes that there is something amiss about early modern mathematics, and about16th- and 17th-century mathematics in particular. By sampling views from some of the most competent, influential historians of mathematics of the last century, we find almost unanimity in pointing out the prevalence of conceptual confusion and lack of rigor in the 17th century. F. Cajori complained that Newton's and Leibniz's principles of the calculus lacked clearness and rigor.[3] D.E. Smith pointed again to 17th-century infinitesimals as unacceptable on 'scientific' grounds.[4] On the other hand and somehow paradoxically,

[3] "The explanations of the fundamental principles of the calculus, as given by Newton and Leibniz, lacked clearness and rigor. ... These objections [namely, Nieuwentijt's] Leibniz was not able to meet satisfactorily." F. Cajori, *A History of Mathematics* (1922, 1sr ed. 1894), p. 218.

[4] "[early writers on indivisibles found areas and volumes] by the summation of an infinite number of infinitesimals. Such a conception of magnitude cannot be satisfactory to any scientific mind, but it formed a kind of intuitive step in the development of the method of integration and undoubtedly stimulated men like Leibniz to exert their powers to place the theory upon a scientific foundation." D.E. Smith, *History of Mathematics*, 2 vol. (1953 (1st ed. 1925)) II: 687.

D.J. Struik and others presented classical cannons as a straitjacket of sorts from which the most creative mathematicians managed to free themselves. These historians suggested that forgetting about classical rigor amounted to intellectual liberation:

> D.J. Struik, *A Concise History of Mathematics* (1948):
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> "[Kepler] broke with Archimedean rigor; ... The proofs of Archimedes, Kepler said, were absolutely rigorous, ... but he left them to the people who wished to indulge in exact demonstrations. [He believed] [e]ach successive author was free to find his own kind of rigor, or lack of rigor, for himself." (p. 129)
>
> "Leibniz's explanation of the foundations of the calculus suffered from the same vagueness as Newton's" (p. 158)

Another influential historian presented Newton's and Leibniz's innovative endeavors in moral terms:

> J.F. Scott, *A History of Mathematics* (1969, 1st ed. 1958):
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> "Leibniz's exposition was no more free from blemish than was Newton's. Both showed similar indecision; neither succeeded in making the fundamental concepts clear. ... Neither Newton nor Leibniz had any scruples about accepting the dubious principle that quantities might become small enough to be neglected. Nowhere is there a hint as to how the gap from finite to infinitely small quantities might be bridged" (p. 159)
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> "Leibniz undoubtedly fashioned his calculus with even less regard for strict logic and rigor than did Newton." (p. 160)

> V.J. Katz, *A History of Mathematics. An Introduction* (1993)
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> "What neither [Newton nor Leibniz] did, however, was establish their methods with the rigor of classical Greek geometry, because both in fact used infinitesimal quantities." (p. 472)

Finally I quote from what is likely the most influential history of mathematics textbook in the last decades, which stresses Struik's view that early modern mathematics advanced through its liberation from logical rigor:

> U.C. Merzbach, C.B. Boyer, *A History of Mathematics* (2011, based on Boyer's 1968 classic):

> "**Rigor and Progress**. ... It is customary to place much of the blame for the **backwardness** in analysis [in England during the 18th century] on the supposedly clumsy method of fluxions ... but such a view is not easily justified. ... Nor is it fair to place the blame for British **geometric conservatism** largely on the shoulders of Newton. ... Perhaps it was an **excessive insistence on logical precision** that had led the British into a narrow geometric view. ... On the Continent, on the other hand, the feeling was akin to the advice ... d'Alembert is said to have given to a hesitating [about the soundness of the calculus] friend: "Just go ahead and faith will return"." (**emphasis added**, p. 381)

Letting aside the fact that most historians criticize on logical grounds the Newtonian fluxional calculus as much as the Leibnizian differential calculus, what is interesting in this quotation is that "an excessive insistence on logical precision" is claimed to lead to stagnation and backwardness.

If we take the foregoing views seriously, they bring trouble for the self-referential image of mathematics that I have just mentioned — namely for the view that mathematical truth results but from just deductions grounded on clear and commonly accepted principles. If just deductions and clear principles were not guaranteed (and that is what is usually meant by non-rigorous thinking), then it is hard to see how a minimum of mathematical unity and coherence could survive. But we know that it did survive. Hence it cannot be that mathematics turned into a field in which more or less lousy thinking was tolerated—or even encouraged, as some of the foregoing quotations suggest. It cannot be that we use "lack of rigor" as an analytical tool to approach early modern mathematics. Can mathematics be mathematics if it lacks rigor?

What I intend to argue now is that what counts as mathematically consistent, or acceptable in mathematical argument, is negotiable. To be precise, I aim to show how it was indeed openly negotiated in a celebrated episode. I want to present a case study in which 17th- and 18th-century mathematicians renegotiated their methodological requirements to make room for new objects they were introducing into mathematical practice. This is the well-known case of infinitesimals, or to put it in 17th-century terms, the infinite divisibility of extension (IDE).

Infinitesimals

In the 17th-century the existence of infinitesimals was discussed as a thesis about the infinite divisibility of extension, or to be more precise about whether extension was infinitely divisible *actually* as well as potentially. The discussion seems to have been prompted in a mathematical context, namely around the status of Cavalieri's indivisibles. Nonetheless it soon enlarged its focus to include physical extension and to apply to space and material bodies along with geometrical lines, surfaces, and bodies. In a sense such discussions were the culmination of a long philosophical tradition that questioned the composition of the continuum. However, in the second half of the 17th century the discussions about infinitesimals (the objects that came to occupy the place of indivisibles) gave a twist to that tradition. I want to emphasize that it became primarily a natural philosophical discussion rather than a simple mathematical or logical technicality. It had wider intellectual implications, and for that reason the discussion was of the outmost importance to Boyle and eventually to Berkeley and to Hume.

It is unfeasible to summarize here all the arguments and counterarguments crossed in the composition of the continuum discussions, and I will restrict myself to point out what was original in them in the second half of the 17th-century. Isaac Barrow was one of the firsts authors who took infinitesimals as a new way out of the old continuum quandary. As far as I know he was the first who provided detailed, systematic arguments in favor of the composition of any finite line by infinitely many infinitely small parts, or, what was assumed to be tantamount, in favor of the existence of an *actual* infinite number of parts in any finite line. This was published in 1683 in his *Lectiones mathematicae*, gathering his lectures in Cambridge in the mid 1660s.[5]

Barrow's discussion falls in three sections. The first contains a good many arguments against an atomistic view of extension, mostly a rehearsal of the classical and Scholastic arguments showing that lines cannot be assumed to be made up of points; nor surfaces of lines; and so on.[6] The second section brings out the difference between classical indivisibles and infinitesimal parts—for which he does not have a proper or specific name. (Parenthetical remark: the word 'infinitesimal part', or 'infinitesimal' for short, seems to have been first used by Nicolas Mercator in the 1660s — Leibniz credited him for that. Yet the word does not seem to gain currency until the turn of the 18th century.) To refer to infinitesimal lines and surfaces Barrow variously used *lineola* (translated as *linelets*), and different circumlocutions such as “parallelograms of a very small, non-considerable (*inconsiderabilis*) ... height”. Barrow was not original in describing infinitesimals as being homogeneous with the continuum of which they are parts, and in stressing that infinitesimals can be further divided and compared to each other. Such notions had been previously introduced and used in mathematical argument

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[5] I. Barrow, *Isaaci Barrow, mathematicae professoris lucasiani Lectiones habitae in scholis publicis Academiae Cantabrigiensis, Annis Domini 1664, 1665, 1666* (London, 1683) = *The Mathematical Works of Isaac Barrow*, W. Whewell ed. (Cambridge, 1860), pp. 23-414. Copies of the book bearing the dates 1684 and 1685 do exist, but I do not know if they respond to different editions or printings. Variations in the title words may suggest a new edition was brought out at least in 1685: *Isaaci Barrow lectiones mathematicae XXIII in quibus principia matheseos generalia exponuntur, habitae Cantabrigiae a. D. 1664, 1665, 1666. Accesserunt ejusdem lectiones IV, in quibus theoremata et problemata Archimedis de sphaera et cylindro methodo analytica eruuntur.* In the 18th century it was translated into English as *The usefulness of mathematical learning explained and demonstrated : being mathematical lectures ... by Isaac Barrow*, John Kirkby, trans. (London, 1734). Barrow's arguments about indivisibles and infinitesimals, in what they differ, and why the composition of the continuum by infinitesimals is acceptable are mostly presented in Lecture IX.

[6] Barrow, *Mathematical Works*, 139-142.

at least by Torricelli, Pascal, and Wallis, although in these authors we do not find elaborate arguments about the nature and status of infinitesimals.[7]

Finally, Barrow's third section sustains the thesis that a "finite magnitude can have an infinity of parts". The core of his argumentation is to be found in some classical paradoxes against the existence of the actual infinite, which he counters. Barrow uses the addition of mathematical series of infinite terms that yield a finite quantity to counter the argument (which he attributes to Epicurus) that it is unintelligible to claim that a line is made up of an infinite number of parts. Infinite additions in mathematical series, Barrow points out, show that there is no contradiction in assuming a line having infinite parts. Next, he faces the paradox that the existence of an actually infinite magnitude leads to assume that some infinites are greater than others, which had traditionally been dismissed as absurd on the assumption that all the infinites are equal. Interestingly Barrow can make use here of a notion not available to classical and medieval thinkers, infinite astronomical space. He imagines an infinite straight line actually extended in space, which provides him with a clear image of notions that were formerly supposed to be absurd. That line would show without inconsistency an infinite number of feet and another infinite number of any other units, and thus the existence of infinites greater than other infinites, or "infinites within infinites":

> for if a right line be supposed to be extended infinitely in space, ... it will doubtless contain an infinite number of feet, an infinite number of paces, and an infinite number of furlongs.[8]

Barrow's discussion, delivered in Cambridge in 1665, was almost certainly attended by Newton. It is therefore interesting to find a similar argument in 1693 in one

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[7] Barrow, *Mathematical Works*, 143. On former adumbrations of the idea of infinitesimal, see V. Jullien, *Seventeenth-Century Indivisibles Revisited* (Berlin, 2015), 105-136, 177-210, 211-248.

[8] Barrow, *Mathematical Works*, 144-148; quotation on p. 146.

of Newton's letters to Bentley. Infinite divisibility entails no logical contradiction, he argued, because not all infinites are equal. For instance, the infinite number of parts in an inch is greater than the number of such parts in a foot: "tho' there be an infinite Number of infinite little Parts in an Inch, yet there is twelve times that Number of such Parts in a Foot".[9]

As was customary in these discussions, Barrow was to close it with a cautious note, by recognizing that his thesis was hard to understand, and by pointing to the limited capacity of the human mind. Nonetheless he stressed that the truth of the "indefinite division" of magnitude was proved by "evident tokens" and supported by "many strong arguments", even though he recognized that we were not able "to comprehend how this indefinite division can be performed". I would like to stress that most of Barrow's preceding arguments as well as the main thrust of his line of reasoning will appear with slight or cosmetic variations in other influential expositions of the infinite division of extension up to the first decades of the 18th century.

One such exposition appears prominently in John Keill's widely known *Introduction to Natural Philosophy*, containing his Oxford lectures. The book was first published in Latin in 1702, then in English in 1720 and many times reprinted in both languages up to the 1750s.[10] Keill (1671-1721) was Newton's protégé, a notorious figure in the quarrel between Newton and Leibniz, an influential Newtonian teacher (Desaguliers was one of his students), and Savilian Professor of Astronomy since 1712. Keill's lessons are intended to provide geometrical grounds for the study of motion and the "true" natural philosophy (meaning Newtonian rather than Cartesian). The first

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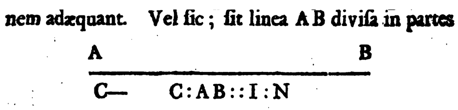
[9] Newton to Bentley, 17 January 1693, in *Newton. Philosophical Writings*, A. Janiak, ed. revised ed. (Cambridge, 2014), 125.

[10] K. Keill, *Introductio ad veram physicam, seu Lectiones physicæ* (Oxford, 1702; further editions in 1705, 1715, 1719, 1741). It was published in English as *An introduction to natural philosophy, or, Philosophical lectures read in the University of Oxford, Anno Dom. 1700* (London, 1720; further editions in 1726, 1733, 1745, 1758).

lectures introduce the general properties of bodies, place, and time, and Keill devoted two of them (out of 16 in the book) to the infinite divisibility of magnitude. As I have just mentioned, the similarities with Barrow's presentation, both in detail and in the overall organization of the argument are striking, and the more so if account is taken of the 40 years gap between the two sets of lessons. I shall come back to this feature that I think needs be stressed, the consistency in methodic attempts (even if widely separated in time) for dealing with what may appear now to be a non-rigorous, fuzzy topic.

Keill, who freely used the word ‘infinitesimal’, simplified Barrow's many references to classical and Scholastic arguments. On the other hand, he made updated references to mathematical practice as additional arguments in favor of the infinite divisibility of extension, including Torricelli's infinitely long solid of finite volume.[11] An argument that gained currency in the last decades of the 17th century was linked to algebraic representations of the infinite. Very early, in the 1650s, Wallis introduced the symbol ∞ for an infinite number and represented an infinitesimal by 1/∞, but it remained an idiosyncratic, Wallisian notation. Keill took up the idea and linked it to a dual representation, algebraic as well as geometric, as in his image:



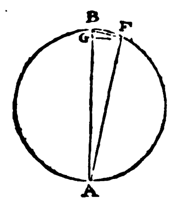
By taking N to represent an “infinite number”, he assumes it is possible to divide the length of any finite arbitrary line, AB, into N parts. Let us call C the resulting part, then Keill concludes by the usual rules that

$$C : AB :: 1 : N$$

[11] Keill, *An introduction to natural philosophy*, p. 36.

which shows that "[C] taken an infinite number of times, will be equal to AB". This, says Keill, proves that any finite quantity can contain infinite parts.[12]

Around 1700, and particularly in the context of Continental mathematicians, these visualizations of infinitesimals gained currency. Bernard Nieuwentijt, famous for his controversy with Leibniz on the calculus, also represented an infinitesimal by the fraction, *b*/*m*, of a quantity *b*, assumed finite, divided by the infinite number *m*.[13] One of the notions most open to question in the first years of the calculus was that of second and higher order differentials, specifically and pointedly criticized by Nieuwentijt on the Continent and by Berkeley in the UK. Keill explicitly justified "infinitesimals of infinitesimals", by the following visualization. Let BA be the diameter of the circle BFA, and let BF represent an infinitesimal of the circle. Remember that the circumference of a circle could be assumed to be a regular polygon with an infinite number of sides. On this assumption, an infinitesimal of the circumference was equated to the side of the infinitely sided regular polygon inscribed in the circle:



Now since BFA is a rectangular triangle and is similar to BGF, by elementary geometry is AB : BF :: BF : BG. That means that the line BG stands to infinitesimal BF in the same proportion as the infinitesimal BF to the finite diameter AB. Hence BG is

[12] Keill, *An introduction to natural philosophy*, p. 35; pages 26-31 and 35-36 are devoted to "invincible" mathematical arguments that, according to Keill, demonstrate that all extension is infinitely divisible.

[13] B. Nieuwentijt, *Analysis infinitorum seu curvilineorum proprietates ex polygonorum natura deductae* (Amsterdam, 1695), 1-4.

an infinitesimal of an infinitesimal.[14] The same argument was still used for the same purpose in d'Alembert defense of second order differentials.[15]

In my view, the most significant difference Keill introduced into Barrow's case for the existence of infinitesimals is not to be found in the new arguments but in the new status he accorded to the infinite divisibility of extension, and to infinitesimals as a consequence. Keill gave the infinite divisibility of magnitude pride of place among the “clear and distinct conceptions” that he imparted to “beginners” concerning the properties of bodies.[16] The IDE was a thesis neither marginal nor tentative, but one that Keill presented as demonstrated by “invincible arguments”. He did not fail to acknowledge the “weakness of the human understanding” when it comes for instance to conceive infinitesimals of infinitesimals, and yet in Keill's account the thesis of infinite divisibility of extension had become so to speak an unquestionable principle of mathematical argument.

In the second half of the 17th century vindications of the infinite divisibility of magnitude, always along the lines sketched here, appear in many telling places. These include John Wallis’ answer to Hobbes's critique of infinitesimals, printed in the *Philosophical Transactions* in 1671, and one letter from Newton to Bentley in 1693, already mentioned.[17] Now, by far the most consequential discussion of infinite divisibility of extension in a general context is found in Boyle's *Discourse of Things Above Reason* (1681).

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[14] Keill, *An introduction to natural philosophy*, 41.

[15] J. d'Alembert, "Differentiel", in *Encyclopédie* (ARTFL Encyclopédie Project, R. Morrisey, Gral Editor), IV (1754), 985, (https://artflsrv03.uchicago.edu/philologic4/encyclopedie1117/navigate/4/4923/, consulted 10 November 2017).

[16] Keill, *An introduction to natural philosophy*, Preface, xi.

[17] *Philosophical Transactions*, 1671, Num. 75, September 18th, pp. 2241-2250; for analysis, see A. Malet, M. Panza, “Wallis on indivisibles”. In V. Jullien, ed. *Seventeenth-century Indivisibles Revisited* (Basel, etc.: Birkhäuser, 2015): 307-346.

As is well know Boyle's *Discourse* —which bears the subtitle, *Inquiring Whether a Philosopher should admit there are any such*— is a sustained analysis of why some crucial notions seem to be out of full reach of the human understanding. In Boyle's concerns, the logical difficulties about the idea of god and other mysteries of the Christian faith were paramount. What is perhaps not so well known is that the issue of infinite divisibility of extension is like the backbone of the arguments Boyle deployed in his *Discourse*.[18] In it, the infinite divisibility of magnitude plays a central role in different ways. First, it provides the most important example of a non-theological notion that is above reason. As such it reappears constantly in Boyle's pages. Furthermore, crucial in Boyle's sustained argument is that in his view the infinite divisibility of extension peculiarly combines mathematical certainty with apparently absurd consequences that seem to derive from it.

Boyle's central point is the following dilemma. On the one hand, he says, geometricians teach us that the ID of E is mathematically demonstrable — and Boyle never doubts or questions the geometrician's claim. On the other, this claim has absurd consequences. Therefore, either we reject the infinite divisibility of extension, which is an inference legitimately drawn from mathematical truths, or we admit it and with it conclusions apparently absurd or at least incomprehensible.[19] The dilemma is important to Boyle because it helps him present *in the same terms* the dilemma posed by sacred notions like the nature of god or of the human soul:

> the most ... subtle sort of Speculators, Metaphysicians, and Mathematicians ... confess themselves quite baffled by the unconquerable difficulties they met with not only in ... the nature of God,

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[18] Surprisingly, the last major study of Boyle's *Discourse*, J. Wojcik's *Robert Boyle and the Limits of Reason* (Cambridge, 1997), is silent about Boyle's multiple references to infinite divisibility.

[19] R. Boyle, *Discourse of Things Above Reason, Inquiring Whether a Philosopher should admit there are any such* (London, 1681), 13-14.

> or of the human Soul, but in the nature of what belongs in common to the most obvious Bodies in the World, and even to the least portion of them: ... [namely] that famous controversie, Whether or no a continued quantity ... be made up of indivisibles.[20]

Boyle is saying that the nature of god and of the human soul poses the same intellectual problems, the same kind of rational doubts, as the nature of ID or of infinitesimals. He deals with these doubts in a positive way. He is not advocating that humans should renounce to use their limited understanding, on the contrary. He is only warning them about the need for caution and intellectual modesty. Above all he is setting forth an interpretive principle, namely, that, if we are lead by clear and legitimate deductions to some conclusion or result that assures us of the existence of some thing ("that some things are", to use Boyle's words), then we must accept the thing (or things), even if we do not fully understand them. This principle is reinforced by Boyle's claim that the deductive ability of the mind is more powerful than its ability to frame clear and distinct ideas, or that humans are better at inferring propositions than at "penetrating the nature of things".[21]

Boyle's point is one of theology, but his implicit argument is that in mathematics we accept things, even if we do not fully understand them, when we are lead by clear and legitimate deductions to them. Let me stress Boyle's eccentric position vis-à-vis the mathematical community, which I think makes him a relevant witness on the logical status of the infinite divisibility of extension. Not being himself a mathematician, his views about the nature of mathematical certainty are all the more representative of what was commonly accepted in English philosophical society. Boyle was just appropriating

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[20] Boyle, *Discourse of Things Above Reason*, 24-25.

[21] Boyle, *Discourse of Things Above Reason*, 71.

himself of uncontroversial views about mathematics to make a controversial theological point.

This important qualification about the nature of mathematical truth also appears in other contemporary authors. In John Wilkins' bestseller, *Principles and Duties of Natural Religion*, mathematical certainty is presented as on a par with, but not above, physical or experimental certainty.[22] Yet probably David Hume's is the most authoritative and influential voice that used mathematics to illustrate the limits of human reason. In his concluding section to *An Enquiry Concerning Human Understanding* (1748), Hume claimed space and time are good examples that abstract ideas lead to absurd conclusions:

> The chief objection against all *abstract* reasonings is derived from the ideas of space and time; ideas, which ... afford principles, which seem full of absurdity and contradiction. [stress in the original]

And he goes on explicitly pointing to the infinite divisibility of extension:

> No priestly *dogmas*, invented on purpose to tame and subdue the rebellious reason of mankind, ever shocked common sense more than the doctrine of the infinitive divisibility of extension, with its consequences; as they are pompously displayed by all geometricians and metaphysicians, with a kind of triumph and exultation.

Notice that Hume is here conflating geometricians and metaphysicians, which is essentially a self-serving way to put him au dessus de la mêlé, or over and above dogmatic thinkers who believed in incontrovertible certainties. As we shall see below, this is misleading, since mathematicians tried hard to separate mathematical discussions from metaphysical contexts. Then Hume goes on to exploit the two attributes of the

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[22] J. Wilkins, *Of the Principles and Duties of Natural Religion* (London, 1675), 5, 8-9; this highly popular work was many times reprinted up through the 1730s.

infinite divisibility of extension, that it is mathematically sound *and* that it has absurd consequences:

> A real quantity, infinitely less than any finite quantity, containing quantities infinitely less than itself, and so on *in infinitum*; this is an edifice so bold and prodigious, that it is too weighty for any pretended demonstration to support, because it shocks the clearest and most natural principles of human reason. But what renders the matter more extraordinary, is, that these seemingly absurd opinions are supported by a chain of reasoning, the clearest and most natural; nor is it possible for us to allow the premises without admitting the consequences.[23]

And so Hume turns mathematical certainties in support for his sceptical philosophy of knowledge:

> The demonstration of these principles [meaning results that support infinite divisibility] seems as unexceptionable as that which proves the three angles of a triangle to be equal to two right ones, though the latter opinion be natural and easy, and the former big with contradiction and absurdity.
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> (...)
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> So that nothing can be more sceptical ... than this scepticism itself, which arises from some of the paradoxical conclusions of geometry.[24]

Hume's appropriation of the infinite divisibility of extension, and the outstanding role he gave to it point to this notion as a well-established, familiar feature in the contemporary British intellectual context. Hume is not entering into the mathematical debate, but takes it for granted that mathematicians belief in infinite divisibility, which in itself is evidence of the weight the thesis had. It reveals that infinitesimals and the

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[23] The three last quotes make a long paragraph in the concluding section of D. Hume, *An Enquiry Concerning Human Understanding*, in *The Philosophical Works*, 4 vol. (London 1882), IV: 1-135, p. 128.

[24] Hume, *An Enquiry Concerning Human Understanding*, 128-129.

infinite divisibility of extension were topics of general import, by no means restricted to narrow circles of specialized mathematicians. Hume's use of the debate is revealing of the logical tensions undermining the debate about infinite divisibility, but also highlights the wide contemporary appeal of the debate. It was not a logical discussion merely appealing to mathematicians, which suggests how open to general debate where questions that concerned the foundations of mathematics.

Let us now consider Berkeley's views from this perspective. At variance with Boyle's and Hume's case, whose concerns with infinitesimals have been largely ignored by mathematical historians, this is not the case with Berkeley's notorious attacks on the foundations of the fluxional calculus. However, Berkeley's criticism of the obscurity of the fundamental notions of the calculus (including infinitesimals, fluxions, and Newton's first and ultimate ratios) and of the demonstrations of key results are often presented out of context, as the quirky reaction of an idiosyncratic philosopher, without taking into account that Berkeley's *Analyst* (1734) falls within a long, well established tradition in which infinitesimals intermix with larger philosophical and theological issues.

Berkeley set forth a line of argumentation that strongly reminds me of Jonathan Swift's ridiculing mathematicians and natural philosophers in Book 3 of *Gulliver's Travels*. This was published in 1726, a few years before *The Analyst*. Both authors are deeply concerned not by the results and methods of the new mathematics and experimental philosophy but by the ever-growing moral and intellectual authority society was bestowing on mathematicians and natural philosophers. The two Christian authors attack Newtonianism and its attending mathematics, the calculus, by trying to undermine its growing prestige as a universal intellectual panacea. Many Newtonians presented Newtonianism as a powerful moral and intellectual guide, one that may be advantageously compared to traditional religious authorities. In reaction to this, Berkeley aimed to subvert the prestige of mathematics. By taking seriously the notion of infinitesimal and its obscurities —which he called the 'metaphysics' of the calculus— and the problematic way in which the were handled — the 'logic'—, Berkeley aimed, first, at weakening mathematics's credit as a reliable source of truths; and secondly, at countering the prestige of mathematics as the best mental exercise for the education of the mind.

Berkeley's criticism of the "logic" and "metaphysics" of the new analysis, allowed him to accuse mathematicians of poor reasoning and to deny to them any intellectual superiority beyond their own subject matter. Addressing the text to the analysts, Berkeley says

> It should seem therefore ... that your conclusions are not attained by just reasoning from clear principles; and consequently that the employment [use] of modern analysis ... doth not habituate ... the mind to apprehend clearly and infer justly; and consequently, ... you [the analysts] have no right to dictate [to teach how to think rightly] out of your proper sphere, beyond which your judgment is to pass for no more than that of other men.[25]

Berkeley contrasted the new notations and symbolic languages (Continental as well as Newtonian) and their well-fixed algorithms with the weakness attending the conceptual content of the notions they represented. The symbolization is "clear and distinct", but behind it there is emptiness and confusion:

> But if we remove the veil and look underneath [the new symbols], if ... we set ourselves attentively to consider the things themselves which are supposed to be expressed or marked thereby, we shall discover much

[25] G. Berkeley, *The Analyst; or, a Discourse Addressed to an Infidel Mathematician. Wherein It is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith* (London, 1734), §49.

> emptiness, darkness and confusion; nay, ... direct impossibilities and contradictions.[26]

Analysts seem to mistake algebraic computation for thinking, but a skillful analyst is not necessarily "a man of science and demonstration", nor are "tedious calculations in algebra and fluxions" likely to be the best method to improve the mind.[27] According to Berkeley, the mathematicians' neglect of metaphysics and their muddled thinking are linked to their belief in the infinite divisibility of extension, a point he already made in 1710 in his *Principles of Human Knowledge*. Berkeley presents the idea at variation with the views above. He points out that infinite divisibility is a thesis that is neither proved as ordinary theorems are, nor it is identified as a postulate — and yet no mathematician ever questions it. Berkeley claims the thesis to be plainly absurd and must therefore be invalidated; this would clear geometry from contradictions that offend human reason:

> The infinite divisibility of finite extension, though it isn't explicitly asserted either as an axiom or as a theorem in the elements of geometry, is assumed throughout it, and is thought to have so inseparable and essential a connection with the principles and proofs in geometry that mathematicians never call it into question. This notion is the source of all those deceitful geometrical paradoxes that so directly contradict the plain common sense of mankind, and are found hard to swallow by anyone whose mind is not yet perverted by learning. ... So if I can make it appear that nothing whose extent is finite contains innumerable parts, or is infinitely divisible, that will immediately •free the science of geometry from a great number of difficulties and contradictions that have always been thought a reproach to human reason, ...[28]

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[26] Berkeley, *The Analyst,* §8; see also §36.

[27] Berkeley, *The Analyst,* §33, §38.

[28] G. Berkeley, *Treatise Concerning the Principles of Human Knowledge* (Dublin, 1710), §123.

Berkeley additional remark is that such a wrong idea is widely accepted because of the power of custom and education. The infinite divisibility of extension, he says, gains the assent of a mind by getting the mind familiar with it little by little. Thus it acquires the status of *principle*, and then whatever is deduced from it is also believed:

> Ancient and rooted prejudices do often pass into principles: and those propositions which once obtain the force and credit of a *principle*, are not only themselves, but likewise whatever is deducible from them, thought privileged from all examination. And there is no absurdity so gross, which by this means the mind of man may not be prepared to swallow.[29]

In the follow-up of the *Analyst*, *A Defence of Free-Thinking in Mathematics* (1735), Berkeley refines this idea by stressing the powerful effect education has on tender minds, particularly when it concerns the basics of knowledge. Young people do not usually question the principles of what they learn, on the contrary they rather trust texts and people whom they belief to be authorities. This is, according to Berkeley, what eventually makes ‘evident’ ideas that are far from being so:

> Men learn the elements of science from others: and every learner hath a deference more or less to authority, especially the young learners, few of that kind caring to dwell long upon principles, but inclining rather to take them upon trust: and things early admitted by repetition become familiar: and this familiarity at length passes for evidence.[30]

Berkeley's claims about the general acceptance of the infinite divisibility of extension and infinitesimals were largely accurate in 1710, when his *Principles of Human Knowledge* was first printed. By the 1730s, however, important mathematicians were expressing doubts about the existence of true infinitesimals. Berkeley's strongest

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[29] Berkeley, *Principles of Human Knowledge*, §124.

[30] G. Berkeley, *A Defence of Free-Thinking in Mathematics* (1735), §21.

opposition to mathematicians was their refusal to seriously engage in clarifying the basic notions of the calculus. Using the language of the time, Berkeley criticized them for refusing to engage in the “metaphysics” of mathematics. In fact the "analysts", says Berkeley, were dismissing his (Berkeley's) critiques by just labeling them “metaphysical”. We have evidence indeed that from the second half of the 17th century on, in most discussions involving definitions of new mathematical objects tensions appeared about how to characterize the new objects, particularly when they could be introduced symbolically (as $\sqrt{(-1)}$). This tension was expressed in terms of how much “metaphysics” was needed to properly define these objects, or the basic notions of mathematics generally. The general trend, as we shall see now, was to sideline "metaphysical" considerations.

The great metaphysician, Leibniz, was not interested in clarifying the definition of his "differences" or "differentials", a discussion that he said belonged to the “metaphysics” of his infinitesimal analysis. More generally, 18th-century mathematicians who concerned themselves with the foundations of the calculus termed their endeavors an exercise "in metaphysics". It must be stressed that from an 18th-century perspective these words only highlight the peripheral status of the question. Most mathematicians focused on the symbolic rules for handling differentials and left the controversial notions (differentials, higher order differentials, etc.) poorly defined. Berkeley, on the contrary, claimed that only metaphysics "can open the eyes of mathematicians" and take them out of the difficulties. He believed that there is a *philosophia prima* or transcendental science superior to mathematics that analysts should learn.[31]

[31] Berkeley, *The Analyst*, queries 51, 49.

Infinitesimals in the 18th century

In Leibniz's hands infinitely small quantities were call differentials and incorporated into a powerful algebraic symbolization. The literature on what Leibniz thought about the nature of differentials is huge and controversial and no consensus seems to be in sight.[32] In any case, a few points of agreement seem to be available. First, in public statements he greatly equivocated about the nature of infinitely small quantities, particularly in correspondence with his followers. From 1676, when Leibniz first claimed that "two quantities [are] equal if their difference can be made ... infinitely small" (*De quadratura arithmetica*); to his permanent emphasis on the (never properly defined) "law of continuity"; to the transcendental law of homogeneity (already mentioned in his seminal article of 1684, "Nova methodus"); to his notorious fictionalist claims; to his famous letter to Varignon of 20 March 1702 (published in the *Journal des Sçavants*), Leibniz set forth contrasting views about infinitesimals. In the 1702 letter just mentioned, he both stated that "mathematical analysis" needed not to rest upon "metaphysical controversies", and that in any case in nature there were no "true" infinitely small lines. Then, to avoid these "subtleties" (*subtilités*), he suggested to explain infinitesimals by things that are incomparably small —like a particle of magnetic fluid, which is incomparably small to a grain of sand, which is incomparably small against the earth globe, and so on. These "incomparable things", Leibniz added,

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[32] M. Katz, Sherry, "Leibniz's infinitesimals: their fictionality, their modern implementations, and their foes from Berkeley to Russell and beyond", *Erkenntnis* 78 (2013): 571-625; Knobloch, ***; D. Rabouin, N.B. Goethe, P. Beeley, "The Interrelations Between Mathematics and Philosophy in Leibniz's Thought", in N. B. Goethe, P. Beeley and D. Rabouin (eds.), *G.W. Leibniz, Interrelations between Mathematics and Philosophy*, Archimedes Series n. 41, Springer Verlag, 2015, pp. 3-21; "Leibniz's Rigorous Foundations of the Method of Indivisibles or How to Reason with Impossible Notions", in V. Jullien (éd.), *Seventeenth-Century Indivisibles Revisited* (Berlin: Birkhäuser, 2015), pp. 347-364; Richard T. W. Arthur, "Leibniz's syncategorematic infinitesimals", *Arch. Hist. Exact Sci*. (2013) 67:553–593; D. Jesseph, "Leibniz on the elimination of infinitesimals: strategies for finding truth in fiction", in N. B. Goethe, P. Beeley and D. Rabouin (eds.), *G.W. Leibniz, Interrelations between Mathematics and Philosophy*, Archimedes Series n. 41, Springer Verlag, 2015, pp. 123-157.

can be understood as one wishes, either as "true" infinites, or as quantities that can be safely not taken into account because the error is so small. Finally, he still suggested to Varignon a third way answering criticisms against the calculus by likening infinitesimals to imaginary roots. Infinitesimals may be understood to be ideal entities (*notions idéales*), as √(-3), or spatial dimensions above three, or powers with exponents that are no ordinary numbers. They are useful and in fact necessary entities that allow mathematical arguments that otherwise may not be performed—even if they do not exist as true mathematical objects.[33]

Throughout the 18th century Leibniz was acclaimed as the founder of the infinitesimal calculus, sometimes on a par with Newton, sometimes putting him above Newton because of his superior notation. In any case he was criticized (in d'Alembert's *Encyclopédie* article, "Differentiel", and also by Lazare Carnot at the end of the century, among others) for his deficient "metaphysics", pointing to Leibniz's equivocations about infinitesimals. D'Alembert suggested that Leibniz might have discovered the calculus on the belief that actual infinitesimals do exist and then, when facing strong criticism, he doubted about his original "metaphysics".[34] As suggested above, Leibniz scholars still debate right now how to understand Leibniz's many conflicting statements about infinitesimals. On the other hand, there is little doubt that up to the 1730s most of his early followers did belief in the existence of true infinitesimals.[35]

Perhaps the most philosophically ambitious project to build up a rigorous geometry of infinite and infinitesimal quantities was Bernard de Fontenelle's (1657-

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[33] "Extrait d'une lettre de M. Leibnitz à M. Varignon", *Journal des Sçavants* (1702), 183-186.

[34] J. d'Alembert, "Differentiel", in *Encyclopédie* (ARTFL Encyclopédie Project, R. Morrisey, Gral Editor), IV (1754), 985-989 (https://artflsrv03.uchicago.edu/philologic4/encyclopedie1117/navigate/4/4923/, consulted 10 November 2017). L. Carnot, *Réflexions sur la métaphysique du calcul infinitésimal* (Paris, 1970), p. 22 (the first, shorter edition appeared in 1797).

[35] M. Horváth, "On the attempts made by Leibniz to justify his Calculus", *Studia Leibnitiana* 18 (1986): 60-71.

1757). His substantial *Elements de la Géométrie de l'Infini* (1727) is grounded on the notion of the actual infinite, the same notion that we encountered in Barrow and Keill.[36] Although deeply knowledgeable in mathematics, he was not a mathematician himself. Permanent Secretary of the Académie Royale des Sciences since 1699, Fontenelle played a major role in bringing mathematics and natural philosophy to the ever-growing public interested in them in contemporary France. Fontenelle's theory was taken seriously but criticized by mathematicians such as C. Maclaurin, Buffon, and d'Alembert. Countering Fontenelle took Buffon and d'Alembert to deny the existence of actually infinite quantities, including time and space. The idea of infinity, they argued, was an abstraction, the result of a mental operation that eliminates from the ideas of real things (which things are always finite) the bounds or limits under which we contemplate or apprehend them. Properly speaking, the infinite is just the negation of the finite, is nothing substantial or real in itself, it exists only in the mind but corresponds to nothing in real quantities.[37]

In the mid 18th century D'Alembert markedly stood out against the consideration of infinitely small quantities as true quantities or true mathematical objects. He simply asserted that such kind of quantities could not exist. He also criticized Leibniz's suggestion that infinitesimals could be taken to be incomparably small quantities—like motes of dust compared to the terrestrial globe—since this would introduce inaccuracies and faulty equations in mathematics. His main positive arguments were, first, the calculus is a worthy method, even if its principles are obscure,

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[36] A. Niderst, *Fontenelle* (Paris, 1991); S. Mazauric, *Fontenelle et l'invention de l'histoire des sciences à l'aube des lumières* (Paris, 2007); B. Fontenelle, *Elements de la geometrie de l'infini*, M. Blay, A. Niderst, eds. (Paris, 1995), with an interesting Preface by M. Blay; J.B. Shank, "Fontenelle's calculus. The cultural politics of mathematics in Louis XIV's France", in *Arts of calculation: quantifying thought in early modern France*, D. Glimp, M.R. Warren, eds. (New York, 2004), 93-114.

[37] J. d'Alembert, "Infini", in *Encyclopédie* (ARTFL Encyclopédie Project, R. Morrisey, Gral Editor), VIII (1765), 703 (https://artflsrv03.uchicago.edu/philologic4/encyclopedie1117/navigate/4/4923/, consulted 10 November 2017). G. L. de Buffon, "Preface" to I. Newton, *La méthode des fluxions et des suites infinies*, G. L. de Buffon, trans. (Paris, 1740), viii-x.

because the results are true—in the sense that are all confirmed by traditional rigorous geometry. Secondly, the ordinary talk of "infinitely small quantities" only serves the purpose of shortening and simplifying the reasoning behind the resolution of problems. In fact it is not actually necessary to introduce them, since the differential calculus leads to investigate the *ratio* of such supposedly infinitesimal quantities, and therefore what is really needed is to find the limit of a quotient between finite quantities according to the geometrical configuration of the problem. This leads to d'Alembert's main conclusion. The essential feature of the calculus is the *algebraic* determination of the limit just mentioned, and it is rather useless to otherwise discuss its "metaphysics".[38] This word, which was widely but loosely used in the context of clarifying the foundations of the calculus, usually referred to considerations related to the conceptual meaning of a mathematical notion—as opposed to considerations involving algebraic and algorithmic specifications.

Concluding remarks

Infinitesimals were new objects introduced by creative mathematicians —people like Stevin, Kepler, and Torricelli— because of their usefulness in solving difficult problems. Infinitesimals clashed with accepted methodological standards concerning definitions because they involved the notion of actual infinite, on the mathematical status of which there was no consensus. Moreover, their use in mathematical argument broke accepted rules of deduction, like assuming $x = y$, if $y - x = dx$. They broke standards and rules because of their novelty, or perhaps it is more exact to say that new objects require new rules. I suggest that what we witness in the debates about

[38] J. d'Alembert, "Differentiel", in *Encyclopédie* (ARTFL Encyclopédie Project, R. Morrisey, Gral Editor), IV (1754), 985-988, (https://artflsrv03.uchicago.edu/philologic4/encyclopedie1117/navigate/4/4923/, consulted 10 November 2017).

infinitesimals around 1700 is a process of negotiations about the best rules that could be implemented not to renounce to infinitesimals. The process included some failed attempts before a working consensus was reached, to be broken again in the second half of the 18th century by d'Alembert and Euler.

What the evidence shows is that mathematicians were *not* a small set of peculiar, idiosyncratic minds breaking *individually* accepted rules of logic and standards of rigor. We are not dealing here with independent individuals that decide of their own accord to break convention. It is not intellectual anarchy and even less disregard for logical consistency that we discern behind the discussions about the infinite divisibility of extension. The logical and metaphysical status of infinitesimals concerned a large community of university dons, mathematicians, and philosophers (experimental and otherwise). Infinitesimals (either directly or in terms of the infinite divisibility of extension) were analyzed and discussed in a systematic and rather consistent way, in the sense that we recognize from Barrow to Keill and beyond a body of arguments and counterarguments about which a basic consensus was reached. Since the new rules of reasoning that applied to infinitesimals were a matter of new conventions, it is possible to say that they were socially dependent.

It is remarkable that the thesis of the infinite divisibility of extension was deeply related to philosophical scepticism, and the relationship worked both ways. On the one hand, scepticism drew on the infinite divisibility of extension as Hume's defense of scepticism incorporated the discussions about it to his argumentary. On the other hand, the stabilization of new mathematical objects and results benefited from the intellectual climate shaped by moderate scepticism and from the attendant views of human understanding that stressed its intrinsic limitations. Metaphysics as it was understood in the 16th and 17th centuries was in the wane—as Buffon put it, “metaphysics ... [is] the most misleading science".[39] As presented in Diderot's very short article in the *Encyclopédie*, "Métaphysique", metaphysics was to be understood as a theorization of practical endeavors—painters, poets, or geometers who reflect on their practices do the right kind of metaphysics. Short of that it was useless or worse:

> Quand on borne l'objet de la *métaphysique* à des considerations vuides & abstraites sur le tems, l'espace, la matiere, l'esprit, c'est une science méprisable.[40]

By the turn of the 18th century, along with the emergence of infinitesimal thinking it took place a radical transformation of fundamental mathematical objects (including number, geometrical magnitude, ratio, and proportionality), and they all required new agreements about new definitions and new rules. Enlarging the significance of the relation between mathematics and scepticism just mentioned, the conceptual renewal that took place in mathematics greatly benefited from an intellectual climate in which systems of thought grounded on clear and obvious principles and then deductively build from the bottom up—say, systems à la Descartes or à la Hobbes—such systems, I say, were falling out of fashion as the 18th century advanced. This must have facilitated that for many mathematicians interest in first principles and axiomatic-deductive foundations vanished, while for others that kind of questions became a matter of "metaphysics". As such, questions of foundations became peripheral and largely immaterial to the actual practice of research mathematicians.

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[39] G. L. de Buffon, "Preface", in I. Newton, *La méthode des fluxions et des suites infinies*, G. L. de Buffon, trans. (Paris, 1740), xxvi.

[40] D. Diderot, "Métaphysique", in *Encyclopédie* (ARTFL Encyclopédie Project, R. Morrisey, Gral Editor), X (1765), 440, (https://artflsrv03.uchicago.edu/philologic4/encyclopedie1117/navigate/4/4923/, consulted 10 November 2017).